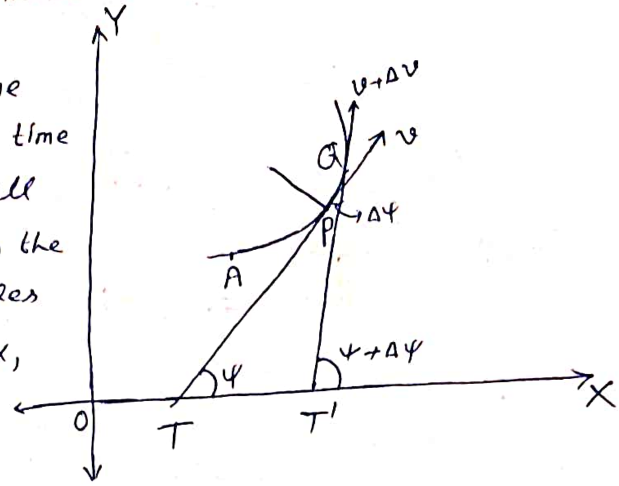
Tangential and Normal Acceleration; [[illegible]]

Q) 1) Find the expressions for tangential and normal accl^n of a particle, moving along a plane curve.

CU - 1996



Let a particle move along a plane curve $y = f(x)$. Let P be the position at time t and Q be the position after a small time Δt. The tangents at P and Q to the curve are PT and QT', making angles ψ and $\psi + \Delta\psi$ with the fixed axis OX, respectively. A is a fixed pt. on the curve. $\widehat{AP} = s$, $\widehat{PQ} = \Delta s$.

Velocity at P = rate of change of displacement at P

$$= \lim_{\Delta t \to 0} \frac{\text{chord } PQ}{\Delta t} = \lim_{\Delta t \to 0} \left\{ \frac{\overline{PQ}}{\Delta s} \cdot \frac{\Delta s}{\Delta t} \right\} \quad [\because \Delta s \to 0 \text{ when } \Delta t \to 0]$$

$$= \lim_{\Delta s \to 0} \frac{\overline{PQ}}{\Delta s} \cdot \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = 1 \cdot \frac{ds}{dt}$$

$\therefore$ Velocity at P, $v = \frac{ds}{dt}$, which is along the tangent TP at P.

There is no component of velocity along the normal at P.

Let $v + \Delta v$ be the velocity at Q, which is along the tangent T'Q.

Let f_s and f_n be the components of acceleration along the tangent and normal at P respectively.

f_s = rate of change of velocity along the tangent at P.

$$= \lim_{\Delta t \to 0} \frac{(v + \Delta v)\cos\Delta\psi - v}{\Delta t} = \lim_{\Delta t \to 0} \frac{-v(1 - \cos\Delta\psi) + \Delta v \cos\Delta\psi}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\{ \frac{-v(1 - \cos\Delta\psi)}{\Delta\psi} \cdot \frac{\Delta\psi}{\Delta t} + \frac{\Delta v}{\Delta t} \cdot \cos\Delta\psi \right\}$$

$$= -v \lim_{\Delta\psi \to 0} \frac{1 - \cos\Delta\psi}{\Delta\psi} \cdot \lim_{\Delta t \to 0} \frac{\Delta\psi}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \cdot \lim_{\Delta\psi \to 0} \cos\Delta\psi \quad \left[\because \Delta t \to 0 \Rightarrow \Delta\psi \to 0\right]$$

$$= -v \times 0 \cdot \frac{d\psi}{dt} + \frac{dv}{dt} \cdot 1$$

or, $f_s = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$.

Again $f_s = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds}$.

$$\therefore \boxed{f_s = \frac{dv}{dt} \text{ or } \frac{d^2 s}{dt^2} \text{ or } v\frac{dv}{ds}}$$

f_n = rate of change of velocity along the perp. direction to the tangent at P.

$$= \lim_{\Delta t \to 0} \frac{(v+\Delta v)\sin\Delta\psi - 0}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\{ \frac{v \sin\Delta\psi}{\Delta\psi} \cdot \frac{\Delta\psi}{\Delta t} + \frac{\Delta v}{\Delta t} \cdot \sin\Delta\psi \right\}$$

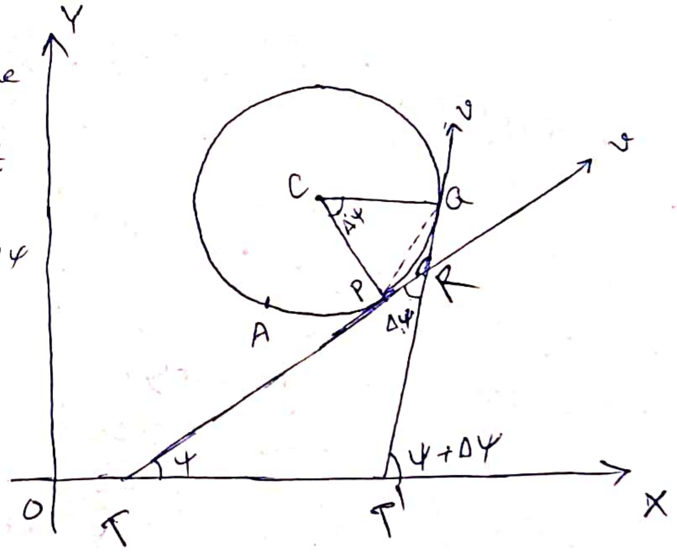
$$= v \lim_{\Delta\psi \to 0} \frac{\sin\Delta\psi}{\Delta\psi} \cdot \lim_{\Delta t \to 0} \frac{\Delta\psi}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \cdot \lim_{\Delta\psi \to 0} \sin\Delta\psi$$

$$= v \cdot 1 \cdot \frac{d\psi}{dt} + \frac{dv}{dt} \cdot 0 = v \frac{d\psi}{dt} = v \frac{d\psi}{ds} \cdot \frac{ds}{dt}$$

$= v^2 \cdot \frac{1}{\rho}$, where ρ = radius of curvature of the curve at P.

$$\boxed{\therefore \; f_n = \frac{v^2}{\rho}}$$

Q)2) Find the tangential and normal components of acc^n of a particle moving along a circle of radius a, with constant speed v.



Let C be the centre of the circle. Let P be the position at time t, Q be the position at time $t+\Delta t$. The tangents at P and Q make angles ψ and $\psi+\Delta\psi$ with the fixed axis OX. A is a fixed pt. on the circle.

$\widehat{AP} = s$, $\widehat{PQ} = \Delta s$

$\because$ The curve is a circle, from geometry $\angle PCQ = \Delta\psi$,

$CP = CQ = a$.

$$\therefore \Delta s = a \cdot \Delta\psi$$

$$\therefore \frac{\Delta s}{\Delta t} = a \cdot \frac{\Delta\psi}{\Delta t}$$

$$\therefore \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = a \lim_{\Delta t \to 0} \frac{\Delta\psi}{\Delta t}$$

or, $\frac{ds}{dt} = a \frac{d\psi}{dt}$.

or, $v = a \frac{d\psi}{dt} = a\omega$,

where $\omega = \frac{d\psi}{dt}$ = angular velocity of the particle about the centre.

$\because$ v is constant, ω is constant.

f_s = tangential component of acc^n

$$= \lim_{\Delta t \to 0} \frac{v \cos\Delta\psi - v}{\Delta t} = -v \lim_{\Delta\psi \to 0} \frac{1-\cos\Delta\psi}{\Delta\psi} \cdot \lim_{\Delta t \to 0} \frac{\Delta\psi}{\Delta t}.$$

$$= -v \cdot 0 \, \frac{d\psi}{dt} = 0.$$

$\therefore \; f_s = 0.$

$f_n =$ normal component of accel^n $= \lim_{\Delta t \to 0} \frac{v \sin\Delta\psi - 0}{\Delta t}$

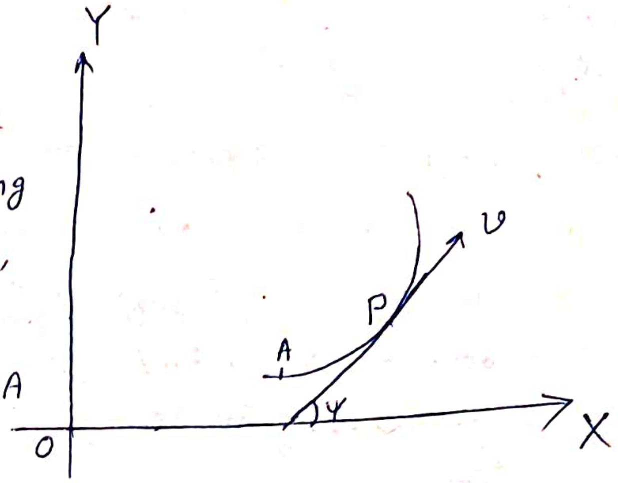
$$= v \lim_{\Delta\psi \to 0} \frac{\sin\Delta\psi}{\Delta\psi} \cdot \lim_{\Delta t \to 0} \frac{\Delta\psi}{\Delta t} = v \cdot 1 \cdot \frac{d\psi}{dt} = v\frac{d\psi}{dt} = v\omega$$

$$= v \cdot \frac{v}{a} = \frac{v^2}{a}.$$

$\therefore \; f_n = \frac{v^2}{a}.$

Q) 3

Ex-2 A particle moves in a curve, so that it's tangential and normal accel^n are equal and the tangent rotates with constant angular vel. Find the path of the particle.



Let P be the position of the particle at time t. v = vel. at P, which is along the tangent to the path making an angle ψ, with the fixed axis OX.

$\widehat{AP} = s$, measured from a fixed pt A on the curve.

$\therefore$ Tangential accel^n $= v\dfrac{dv}{ds}$

Normal accel^n $= \dfrac{v^2}{\rho}$

By condition, $v\dfrac{dv}{ds} = \dfrac{v^2}{\rho}$ or, $\dfrac{dv}{ds} = \dfrac{v}{\rho}$.

or $\dfrac{dv}{v} = \dfrac{ds}{\rho} = \dfrac{ds}{\frac{ds}{d\psi}} = d\psi$

Integrating, $\log v = \psi + \log c$.

or $\log \dfrac{v}{c} = \psi$ $\therefore$ $v = ce^{\psi}$

Let $v = v_0$ when $\psi = \psi_0$. (at $t=0$)

$\therefore$ $v_0 = c\,e^{\psi_0}$ $\therefore$ $v = v_0 e^{\psi - \psi_0}$.

or, $\dfrac{ds}{dt} = v_0 e^{\psi - \psi_0}$ or, $ds = v_0 e^{\psi - \psi_0} dt = v_0 e^{\psi - \psi_0} \dfrac{dt}{d\psi} \cdot d\psi$

Angular velocity of the tangent $= \dfrac{d\psi}{dt} = \omega$ (const) (given)

$\therefore$ $ds = v_0 e^{\psi} \cdot \dfrac{1}{\omega} d\psi$

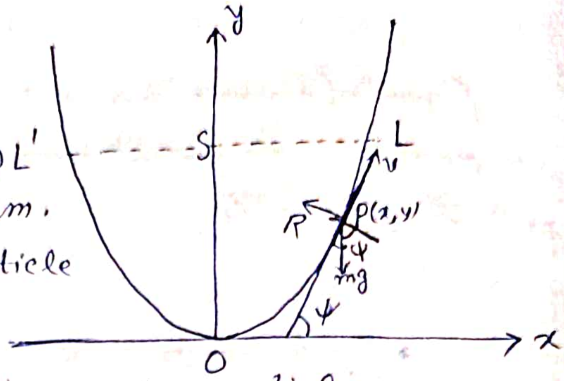
Integrating, $s = \dfrac{v_0}{\omega} e^{\psi} + c_1$

Let $\psi = 0$, when $s = 0$, $\therefore$ $0 = \dfrac{v_0}{\omega} e^{0} + c_1$ or, $c_1 = -\dfrac{v_0}{\omega}$.

$\therefore$ $s = \dfrac{v_0}{\omega} e^{\psi} - \dfrac{v_0}{\omega} = \dfrac{v_0}{\omega}(e^{\psi} - 1)$

This is the intrinsic equation of the path.

Ex-3 ✓ A smooth wire bent in the form of a parabola is fixed with its axis vertical and vertex downwards. A particle of mass m oscillates on the wire comming to rest at the extremities of the latus rectum. Show that the reaction of the wire on the particle when passing through the vertex is 2mg.



Let the equation of the parabolic wire be $x^2 = 4ay$, with respect to horizontal and vertical axes. LSL' is the latus rectum. $P(x,y)$ is the position of the particle at time t.

R = normal pressure of the wire on the particle.

mg = weight of the particle acting vertically downward.

v = vel. at P, which is along the tangent making an angle ψ with x-axis. $\overset{\frown}{OP} = s$.

The equations of motion along the tangential and normal directions are

$$mv\frac{dv}{ds} = -mg\sin\psi \quad \cdots (1)$$

$$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \cdots (2)$$

From (1) $v\frac{dv}{ds} = -g\sin\psi$

or, $v\frac{dv}{ds} = -g\frac{dy}{ds}$

or, $v\,dv = -g\,dy$

Integrating, $\frac{v^2}{2} = -gy + c$

Let the particle starts at L at rest. $\therefore v = 0$, when $y = a$.

$\therefore 0 = -ga + c$ or, $c = ga$.

$\therefore \frac{v^2}{2} = -gy + ga$ or, $v^2 = 2g(a-y)$

When $y = 0$ at O, $v^2 = 2ga > 0$.

$\therefore$ The particle moves to the left of O and again $v = 0$ when $y = a$ at L'.

~~The particle moves to the left~~

$\therefore$ The particle oscillates between L and L'.

from (2), $R = m\frac{v^2}{\rho} + mg\cos\psi \quad \cdots\cdots (3)$

$x^2 = 4ay$, $\therefore 2x = 4a\frac{dy}{dx}$ or, $\frac{dy}{dx} = \frac{x}{2a}$

or, $\frac{d^2y}{dx^2} = \frac{1}{2a}$

$\therefore \rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{\left(1+\frac{x^2}{4a^2}\right)^{3/2}}{\frac{1}{2a}} = 2a\left(\dfrac{4a^2+x^2}{4a^2}\right)^{3/2} = \dfrac{(4a^2+x^2)^{3/2}}{4a^2}$.

$\tan\psi = \dfrac{dy}{dx} = \dfrac{x}{2a}$. $\therefore \sec^2\psi = 1+\tan^2\psi = 1+\dfrac{x^2}{4a^2} = \dfrac{4a^2+x^2}{4a^2}$

$\therefore \cos^2\psi = \dfrac{4a^2}{4a^2+x^2}$ $\therefore \cos\psi = \dfrac{2a}{\sqrt{4a^2+x^2}}$

from (3) $R = m\cdot 2g(a-y)\dfrac{4a^2}{(4a^2+x^2)^{3/2}} + \dfrac{mg\,2a}{\sqrt{4a^2+x^2}}$

At the vertex $x=0,\ y=0$.

~~then~~ Then, $R = 2mga\cdot\dfrac{4a^2}{8a^3} + \dfrac{2mag}{2a} = mg+mg$ ~~$=2mg$~~

$= 2mg$.

Ex-4 A particle moves on the curve $y = a\log\sec\left(\frac{x}{a}\right)$ in such a way that the tangent to the curve rotates uniformly. Prove that the resultant accel^n of the particle varies as the square of the radius of curvature.

Let the tangent at (x,y) to the curve makes an angle ψ with the x-axis and the length of arc is s, measured from the origin to the pt (x,y).

$\therefore \dfrac{d\psi}{dt}$ = angular vel. of the tangent $= k$ (const)

$y = a\log\sec(x/a)$

$\dfrac{dy}{dx} = a\cdot\dfrac{1}{\sec(x/a)}\cdot\sec\left(\frac{x}{a}\right)\tan\left(\frac{x}{a}\right)\cdot\dfrac{1}{a} = \tan\dfrac{x}{a}$.

$\therefore \tan\psi = \tan\dfrac{x}{a}$ $\therefore \psi = \dfrac{x}{a}$ $\therefore x = a\psi$.

$\left(\dfrac{ds}{dx}\right)^2 = 1+\left(\dfrac{dy}{dx}\right)^2 = 1+\tan^2\dfrac{x}{a} = \sec^2\left(\dfrac{x}{a}\right)$

$\therefore \dfrac{ds}{dx} = \sec(x/a)$ $\therefore ds = \sec(x/a)\,dx$

$= \sec\psi\, a\, d\psi$.

$\therefore \dfrac{ds}{d\psi} = a\sec\psi$ or, $\rho = a\sec\psi$.

Now $\dfrac{ds}{dt} = a\sec\psi\dfrac{d\psi}{dt} = ak\sec\psi$

$\therefore \dfrac{d^2s}{dt^2} = ak\sec\psi\tan\psi\dfrac{d\psi}{dt} = ak^2\sec\psi\tan\psi$

Tangential accel^n $= f_s = \dfrac{d^2s}{dt^2} = ak^2\sec\psi\tan\psi$.

f_n = normal accel^n $= \dfrac{v^2}{\rho} = \dfrac{a^2k^2\sec^2\psi}{a\sec\psi} = ak^2\sec\psi$.

Resultant accl^n $= f = \sqrt{(f_s)^2 + (f_n)^2}$

$$= \sqrt{a^2 K^4 \sec^2\psi \tan^2\psi + a^2 K^4 \sec^2\psi}$$

$$= aK^2 \sec\psi \sqrt{1+\tan^2\psi} = aK^2 \sec^2\psi = aK^2 \cdot \frac{\rho^2}{a^2} = \frac{K^2\rho^2}{a}$$

$\therefore \; f \propto \rho^2.$

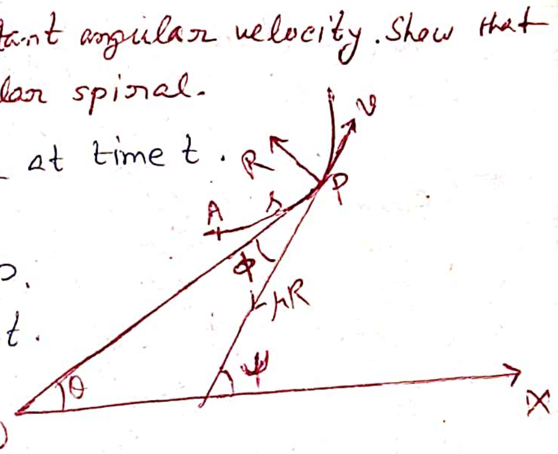
Ex-6 A bead moves along a rough curved wire which is such that it changes its direction of motion with constant angular velocity. Show that the wire is in the form of an equiangular spiral.

Let P be the position of the particle at time t.

R = normal reaction.

v = velocity of the particle at P.

μR = Frictional force along the tangent.



Let the tangent makes an angle ψ with the initial line OX. (Arc AP = s measured from a fixed pt A).

The equations of motion are, $mv\frac{dv}{ds} = -\mu R \cdots (1)$

$$m\frac{v^2}{\rho} = R \cdots\cdots (2)$$

(1) + (2) $\times \mu \Rightarrow$ $m\left(v\frac{dv}{ds} + \frac{\mu v^2}{\rho}\right) = 0$ or, $v\frac{dv}{ds} + \frac{\mu v^2}{\rho} = 0$

or $\frac{dv}{ds} = -\frac{\mu v}{\rho}$ or, $\frac{dv}{v} = -\frac{\mu}{\rho}ds = -\mu d\psi$ $\left[\because \rho = \frac{ds}{d\psi}\right]$

Integrating $\log v = -\mu\psi + \log C_1$ i.e. $\frac{v}{C_1} = e^{-\mu\psi}$

ie: $v = C_1 e^{-\mu\psi}$ or $\frac{ds}{dt} = C_1 e^{-\mu\psi} = \frac{ds}{d\psi}\cdot\frac{d\psi}{dt}$

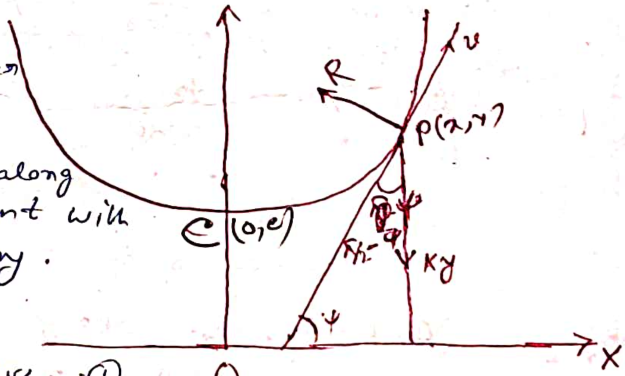
Or, $\frac{ds}{d\psi}\cdot K = C_1 e^{-\mu\psi}$ $\left[\because \frac{d\psi}{dt} = \text{constant} = K\ (\text{say})\right]$

or $ds = \frac{C_1}{K}e^{-\mu\psi}d\psi$

or Integrating, $s = \frac{C_1}{K}\cdot\frac{e^{-\mu\psi}}{-\mu} + C_2 = -\frac{C_1}{K\mu}e^{-\mu\psi} + C_2 - (3)$

From initial conditions C_1 and C_2 can be known. Then (3) is the intrinsic equation of the path of the particle and it represents an equiangular spiral.

Ex-7 A horizontal smooth fixed plane wire has the shape of the catenary $y = c\cosh(x/c)$ and carries a small ring which is attracted towards the directrix with a force proportional to the distance y from the directrix. Show that the ring has S.H.M. and that the normal reaction on it varies as the reciprocal of y^2 [The formulae $s = c\tan\psi$ and $y = c\sec\psi$ may be used.]



Let $P(x,y)$ be the position of the ring at time t, R = normal reaction. The force per unit mass = Ky towards the directrix where K = +ve constant.

Let v be the velocity at P which is along the tangent at P. ψ = angle made by the tangent with OX. C is the lowest pt of the catenary. $\overset{\frown}{CP} = s$, m = mass.

The equations of motion are,

$$mv\frac{dv}{ds} = -mKy\sin\psi \cdots (1)$$

and $m\frac{v^2}{\rho} = R - mKy\cos\psi \cdots (2)$ $\left[\begin{array}{l}(1)\text{ can be written as} \\ m\frac{d^2s}{dt^2} = -mKy\sin\psi \\ \text{or } \frac{d^2s}{dt^2} = -Kc\tan\psi = -Ks\end{array}\right]$

From (1) $v\frac{dv}{ds} = -Ky\frac{dy}{ds}$

i.e. $v\,dv = -Ky\,dy$

Integrating, $\frac{v^2}{2} = -K\cdot\frac{y^2}{2} + \frac{C_1}{2}$

or $v^2 = -Ky^2 + C_1$ or, $\left(\frac{ds}{dt}\right)^2 = -Ky^2 + C_1$

or, $2\frac{ds}{dt}\cdot\frac{d^2s}{dt^2} = -2ky\frac{dy}{dt}$ [diff. w.r.t. t]

or $\frac{ds}{dt}\cdot\frac{d^2s}{dt^2} = -k.c\sec\psi.\frac{dy}{dt}$ [$\because y = c\sec\psi$]

or $\frac{d^2s}{dt^2} = -kc\sec\psi\cdot\frac{dy}{ds} = -kc\sec\psi\sin\psi = -k(\tan\psi)c$

or $\frac{d^2s}{dt^2} = -ks$ [$\because s = c\tan\psi$] - - (3)

This shows that the motion of the ~~particle~~ ring is S.H.M.

Multiplying (3) by $2\frac{ds}{dt}$ and integrating we have,

$$\left(\frac{ds}{dt}\right)^2 = 2\int -ks\,ds + C_2 = -2k\cdot\frac{s^2}{2} + C_2$$

or $v^2 = -ks^2 + c_2$

c_2 can be known from initial condition.

From (2) $R = m\frac{v^2}{\rho} + mky\cos\psi = \frac{m}{\rho}(-ks^2+c_2) + mkc\sec\psi\cos\psi$

$$\therefore R = \frac{m(-ks^2+c_2)}{c\sec^2\psi} + mkc \quad \left[\begin{array}{l} s = c\tan\psi \\ \frac{ds}{d\psi} = c\sec^2\psi = \rho \end{array}\right]$$

$$= \frac{-mkc^2\tan^2\psi}{c\sec^2\psi} + \frac{mc_2}{c\sec^2\psi} + mkc$$

$$= \frac{-mkc^2(\sec^2\psi - 1)}{c\sec^2\psi} + \frac{mc_2}{c\sec^2\psi} + mkc$$

$$= -mkc + \frac{mkc}{\sec^2\psi} + \frac{mc_2}{c\sec^2\psi} + mkc$$

$$= \frac{m}{\sec^2\psi}\left[kc + \frac{c_2}{c}\right] = \frac{mc^2\left[kc + \frac{c_2}{c}\right]}{y^2}$$

$\therefore R \propto \frac{1}{y^2}$, (proved)

Ex-8 A particle is projected horizontally from the lowest pt of a rough sphere of radius a and comes to rest at the lowest pt after describing an arc less than a quandrant; Show that the velocity of projection must be $(\sin\alpha)\sqrt{\frac{2ga(1+\mu^2)}{1-2\mu^2}}$, where α is the angle sustended by the arc at the centre and μ is the coefficient of friction.

'O' is the centre of the sphere, A is the lowest paint,

u = velocity of projection, P the Position at time t.

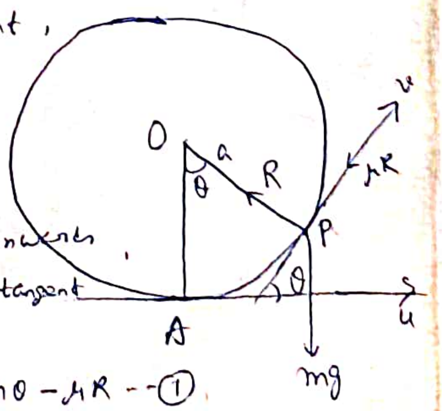
$\angle AOP = \theta$, $\widehat{AP} = s = a\theta$.

v = velocity at P.

R = Normal reaction at P along PO.

mg = weight of the particle, acting vertically downwards.

The tangent at P, makes the angle θ with the tangent at A (From geometry).



The equations of motion are $mv\frac{dv}{a\,d\theta} = -mg\sin\theta - \mu R$ - - (1)

and $m\frac{v^2}{a} = R - mg\cos\theta$ - - - (2)

(1) + (2) × μ, $\quad v\frac{dv}{a\,d\theta} + \frac{v^2\mu}{a} = -g(\sin\theta + \mu\cos\theta)$

or $\frac{1}{2}\frac{d}{d\theta}(v^2) + \mu v^2 = -ag(\sin\theta + \mu\cos\theta)$

or $\frac{d(v^2)}{d\theta} + 2\mu v^2 = -2ag(\sin\theta + \mu\cos\theta) \cdots (3)$

which is a linear equation in v^2.

$$I.F = e^{\int 2\mu d\theta} = e^{2\mu\theta}$$

multiplying (3) by I.F and integrating,

$$v^2 \cdot e^{2\mu\theta} = -2ag\int e^{2\mu\theta}(\sin\theta + \mu\cos\theta)\,d\theta + C_1$$

$$= -2ag\left[\frac{e^{2\mu\theta}}{4\mu^2+1}\left\{(2\mu\sin\theta - \cos\theta) + \mu(2\mu\cos\theta + \sin\theta)\right\}\right] + C_1$$

$$= \frac{-2age^{2\mu\theta}}{4\mu^2+1}\left[3\mu\sin\theta + (2\mu^2-1)\cos\theta\right] + C_1 \quad \cdots (4)$$

$$\boxed{\begin{aligned} \int e^{ax}\cos bx\,dx &= \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} \\ \int e^{ax}\sin bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \end{aligned}}$$

When $\theta = \alpha$, $v = 0$

$$0 = \frac{-2age^{2\mu\alpha}}{4\mu^2+1}\left[3\mu\sin\alpha + (2\mu^2-1)\cos\alpha\right] + C_1 \quad \cdots (5)$$

At A, $v = u$, $\theta = 0$

$$\therefore u^2 = -\frac{2ag}{4\mu^2+1}\left[(2\mu^2-1)\right] + C_1 \quad \cdots (6)$$

(6) − (5) $\Rightarrow$ $$u^2 = -\frac{2ag}{4\mu^2+1}\left[2\mu^2-1 - \{3\mu\sin\alpha + (2\mu^2-1)\cos\alpha\}e^{2\mu\alpha}\right] \cdots (7)$$

When the particle moves downward from the position of rest at $\theta = \alpha$, the direction of the frictional force changes.

The previous equations (1) − (5) hold for $(-\mu)$ in place of (μ)

$\therefore$ From (4) we get,

$$v^2 \cdot e^{-2\mu\theta} = \frac{-2age^{-2\mu\theta}}{4\mu^2+1}\left[-3\mu\sin\theta + (2\mu^2-1)\cos\theta\right] + C_2$$

When $\theta = \alpha$, $v = 0$

$$\therefore 0 = \frac{-2age^{-2\mu\alpha}}{4\mu^2+1}\left[-3\mu\sin\alpha + (2\mu^2-1)\cos\alpha\right] + C_2 \quad \cdots (8)$$

At A, $\theta = 0$, $v = 0$,

$$\therefore 0 = -\frac{2ag}{4\mu^2+1}\left[-3\mu\sin 0 + (2\mu^2-1)\right] + C_2 \quad \cdots (9)$$

(9) − (8) $\Rightarrow$ $$0 = -\frac{2ag}{4\mu^2+1}\left[(2\mu^2-1) + \{3\mu\sin\alpha - (2\mu^2-1)\cos\alpha\}e^{-2\mu\alpha}\right]$$

ie $$(1-2\mu^2) = \{3\mu\sin\alpha - (2\mu^2-1)\cos\alpha\}e^{-2\mu\alpha}$$

ie $$e^{2\mu\alpha} = \frac{3\mu\sin\alpha - (2\mu^2-1)\cos\alpha}{1-2\mu^2}$$

From (7) $$u^2 = \frac{2ag}{4\mu^2+1}\left[\{3\mu\sin\alpha + (2\mu^2-1)\cos\alpha\}\left\{\frac{3\mu\sin\alpha - (2\mu^2-1)\cos\alpha}{1-2\mu^2}\right\} + (1-2\mu^2)\right]$$

$$= \frac{2ag}{4\mu^2+1}\left[\frac{9\mu^2\sin^2\alpha - (2\mu^2-1)^2\cos^2\alpha + (1-2\mu^2)^2}{(1-2\mu^2)}\right]$$

$$= \frac{2ag}{4\mu^2+1}\left[\frac{9\mu^2 \sin^2\alpha + (2\mu^2-1)^2 \sin^2\alpha}{(1-2\mu^2)}\right] = \frac{2ag}{4\mu^2+1}\left[\frac{\sin^2\alpha\{9\mu^2+4\mu^4-4\mu^2+1\}}{(1-2\mu^2)}\right]$$

$$= \frac{2ag}{4\mu^2+1}\left[\frac{\sin^2\alpha\,(5\mu^2+1+4\mu^4)}{(1-2\mu^2)}\right] = \frac{2ag}{4\mu^2+1}\left[\frac{\sin^2\alpha\{4\mu^2(\mu^2+1)+1(\mu^2+1)\}}{(1-2\mu^2)}\right]$$

$$= \frac{2ag}{4\mu^2+1} \cdot \frac{(4\mu^2+1)(\mu^2+1)\sin^2\alpha}{(1-2\mu^2)} = \sin^2\alpha \cdot \frac{2ag(\mu^2+1)}{1-2\mu^2}$$

$$\therefore\ u = \sin\alpha\sqrt{\frac{2ag(\mu^2+1)}{1-2\mu^2}}, \text{ (proved)}$$